— Exercises —

1. **Application to approximation.** Show that the function f defined by $f(x, y) = exp(sin(x)cos(y))$ is differentiable in any point. Using this, give an approximate value of $exp(sin(3.16)cos(0.02))$. *Hint: Use* $\pi \approx 3.14$ *and* $f(x + h, y + k) \approx f(x, y) + Df(x, y).(h, k)$.

2. **A differentiable function defined on a matrix space.** We identify the space of (n, n)-matrices with $\mathbb{R}^{n^2}$.

   (a) Explain why the function $\det \mathbb{R}^n \to \mathbb{R}$ is differentiable.

   (b) Compute the derivative of

$$\begin{aligned} \det : \quad \mathbb{R}^4 &\to \mathbb{R} \\ \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right] &\mapsto x_1x_4 - x_2x_3. \end{aligned}$$

3. **Differentiability of norms.**

   (a) Prove that a norm on $\mathbb{R}^n$ is never differentiable at 0.

   (b) Find a function defined on $\mathbb{R}^n$ which is Lipschitz but not differentiable.

— Problems —

4. **Study of a family of functions.** Let $\alpha \in \mathbb{R}_{>0}$. We define f on $\mathbb{R}^2$ by $f(0, 0) = 0$ and $f(x, y) = \frac{|xy|^\alpha}{\sqrt{x^2+y^2}}$. Study the continuity and the differentiability of f at $(0, 0)$.

5. **Differentiable and homogeneous of degree** 1 **implies linear.** Let f be a differentiable function on $\mathbb{R}^n$ such that for every $x \neq 0$ and $\lambda > 0$, we have $f(\lambda x) = \lambda f(x)$. Show that f is linear.

6. **Linear maps and Lipschitz functions**. Show that any linear map $\mathbb{R}^n \to \mathbb{R}^p$ is Lipschitz.

7. **Another expression of the tangent space of a graph.** Let ϕ be the function defined on $\mathbb{R}^3$ by $\phi(x, y, z) = x^2 + y^2 + z^2 - 1$. We note A the set $\phi^{-1}(\{0\}) \subset \mathbb{R}^3$.

   (a) We put B for the set $B = \{(x, y, z) \in A \mid z > 0\}$. Show that there exists a function f defined on some subset of $\mathbb{R}^2$ such that $B = \{(x, y, z) \mid z = f(x, y)\}$. Verify that f and ϕ are differentiable in any point.

   (b) Give an equation of the tangent space $T_{(a,b,c)}B$ of B at any point (a, b, c) in B.

   (c) Check that $T_{(a,b,c)}B$ can also be expressed as $T_{(a,b,c)}B = (a, b, c) + \ker D\phi(a, b, c)$.

   (d) If z is equal to 0, is it possible to define the tangent space of $(x, y, z) \in B$?

Proofs or ideas of proof

1. A differentiable function defined on a matrix space. det is differentiable at any point since it is a polynomial function. Moreover, we have...

2. Study of a family of functions. In polar coordinates, we have $f(x,y) = r^{2\alpha-1}|cos(\theta)sin(\theta)|^{\alpha}$. If $2\alpha - 1 > 0$ (i.e. $\alpha > \frac{1}{2}$), f is continuous at $(0,0)$. If $\alpha \leq \frac{1}{2}$, f is not continuous at $(0,0)$ (consider sequences $(\frac{1}{n}, \frac{1}{n})$for example; case $\alpha = 1/2$ may be special). Moreover, for every $\alpha > 0$, f has partial derivatives at $(0,0)$ and $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Then, f is differentiable at $(0,0)$ if and only if $\frac{f(x,y)}{||(x,y)||}$ tends to 0 when $||(x,y)||$ tends to 0. Using polar coordinates, we can show that it is true if and only if $2\alpha - 2 < 0$ ($\alpha < 1$).

3. **Application to approximation.** Compute partials and check f is C^1. We have $f(\pi + h, 0 + k) \approx f(\pi, 0) + h\frac{\partial f}{\partial x}(\pi, 0) + k\frac{\partial f}{\partial y}(\pi, 0) = 1 - h \approx 0.98$

4. **A Lipschitz function which is not differentiable.** Every norm on $\mathbb{R}^n$ is a suitable example. For instance, let $f(x)$ be the euclidean norm of $x \in \mathbb{R}^n$. A direct computation shows that f has no partial derivative in any direction at $(0,0)$ ($f(0 + te_i) = ||te_i|| = |t|\ ||e_i||$). So f is not differentiable at $(0,0)$. However, f is clearly a Lipschitz function.

5. **Differentiability of norms.**
   (a) Same proof as previous exercise applies.
   (b) $||\cdots||_2$ is differentiable on $\mathbb{R}^n \setminus \{0\}$ and $||\cdot||_\infty$ is differentiable on $\mathbb{R}^2 \setminus \{(x,y)/|x| = |y|\}$.

— Problems —

6. Linear maps and Lipschitz functions

7. **Another expression of the tangent space of a graph.**
   (a) If (x,y,z) belongs to A, then we have $z^2 = 1 - x^2 - y^2$. Moreover, if $z > 0$, we obtain that $z = \sqrt{1 - x^2 - y^2}$. Then $f(x,y) = \sqrt{1 - x^2 - y^2}$ is suitable.
   (b) The tangent space of B at (x_0, y_0, z_0) has equation $T_{(x_0,y_0,z_0)}B : \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - (z - z_0) = 0$. A computation shows that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{-x_0}{\sqrt{1 - x_0^2 - y_0^2}} = \frac{-x_0}{z_0}$ and $\frac{\partial f}{\partial y}(x_0, y_0) = \frac{-y_0}{z_0}$.
   (c) We can put the equation of $T_{(x_0,y_0,z_0)}B$ under the form $x_0 x + y_0 y + z_0 z = 1$, i.e $D\varphi(x_0, y_0, z_0)(x, y, z) = 2$. But $D\varphi(x_0, y_0, z_0)(x_0, y_0, z_0) = 2$. So the equation of $T_{(x_0,y_0,z_0)}B$ is equivalent to $D\phi(x_0, y_0, z_0)((x, y, z) - (x_0, y_0, z_0)) = 0$.
   (d) If z is equal to 0, we can do the same computation using x or x instead of z (one of these two numbers is nonzero).